DECIDABLE MODELS†

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ABSTRACT
A model $\mathfrak{A}$ is decidable if $\mathrm{Th}(\mathfrak{A}, a)_{a\in A}$ is recursive. Various results about decidable models are discussed. A necessary and sufficient condition for there to be a decidable saturated model is given.

1. Preliminaries

Capital German letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ will denote relation structures with universe A, B, C respectively. In this paper they will always have a countable universe and a countable number of relations. Further we shall always assume there is an effective enumeration of the universe and a fixed Gödel numbering of the formulas in the associated first order language.

A structure $\mathfrak{A}$ is *decidable* if $\mathrm{Th}(\mathfrak{A}, a)_{a\in A}$ is recursive, that is, if the set of pairs of formulas φ and finite sequences $\bar{a}$ from A, $\{\langle \varphi, \bar{a}\rangle; \mathfrak{A} \models \varphi[\bar{a}]\}$, is recursive. This is stronger than only requiring the relations to be recursive. For example, $\langle \omega, +, \times \rangle$ has recursive relations but it is not decidable. In some papers (e.g. [4]) these are distinguished as *constructive* and *strongly constructive*. Speaking more loosely we shall say $\mathfrak{A}$ is decidable if $\mathfrak{A}$ is isomorphic to a decidable structure.

If $\bar{a}$ is a finite sequence in A then the *type* of $\bar{a}$ is the set of formulas satisfied by $\bar{a}$ in $\mathfrak{A}$. The type is recursive if the set of Gödel numbers of the formulas is recursive. A *list* of recursive types is a set of r.e. indices for the types.

2. The most important fact about decidable structures is the following well known result.

THEOREM 2.1. *A decidable theory has a decidable model.*

The proof consists of noting that for decidable theories the Henkin proof of the completeness theorem is effective. Since most of the results of this paper are

† Dedicated to the memory of Abraham Robinson

proved by variants of the Henkin construction we shall give a more explicit description of it.

Given a consistent theory T in a countable first order language, we enlarge the language by adding new individual constants $c_n (n \in \omega)$. An enumeration $\sigma_m (m \in \omega)$ of the sentences in the enlarged language is chosen such that c_n does not occur in σ_m for $m \leqq n$. The theory T is enlarged inductively as follows:

i) at stage $2n$ either σ_n or $\neg \sigma_n$ is added so as to be consistent with T and the sentences previously added.

ii) if the sentence added at stage $2n$ has the form $\exists x \varphi(x)$ then $\varphi(c_n)$ is added at stage $2n + 1$.

As is well known this defines a complete theory in the enlarged language and a model of T with universe $\{c_n ; n \in \omega\}$.

If T is decidable the preceding may be done effectively and hence:

THEOREM 2.2. *T has a decidable model if and only if T has a decidable extension if and only if T has a decidable completion.*

3. Omitting types

Throughout this section we will assume T is a complete decidable theory.

By a variant of the Henkin construction one may construct models of T omitting any non-principal type (i.e., one not generated by a single formula) of finite sequence. (See, for example, [5].) By noting that this construction can be made effective, we have:

THEOREM 3.1. *i) For any non-principal type there is a decidable model of T omitting it.*

ii) For any recursive list of recursive non-principal types there is a decidable model of T omitting all of them.

Note in i) that if the non-principal type is not recursive then it follows directly from Theorem 2.1 since no decidable model can have a non-recursive type.

The following strengthening of 3.1 appears in Millar [3].

THEOREM 3.2. *For every Σ_2^0 list of non-principal type there is a decidable model of T omitting all of them.*

One expects that the decidable models of T would be among the simpler ones. Leo Harrington [2] considered the prime model and showed:

THEOREM 3.3. *Suppose T has a prime model. A necessary and sufficient condition that it be decidable is that there be a recursive list of the principal types.*

The necessity is obvious but the other direction requires a priority argument. Harrington also showed by example that there is:

i) a decidable theory T with a prime model which is not decidable, and

ii) a decidable prime model which has no recursive list of the generators of the principal types.

Finally we have:

THEOREM 3.4. *If T does not have a decidable prime model then T has an infinite number of non-isomorphic decidable models.*

Note that we do not assume T has any prime model.

PROOF. By Theorem 2.1 T has a decidable model $\mathfrak{A}_0$. If $\mathfrak{A}_0$ is not prime it realizes some non-principal type p_0. By Theorem 3.1 there is a decidable $\mathfrak{A}_1$ omitting p_0. If $\mathfrak{A}_1$ is not prime it realizes a non-principal type p_1. There is then a decidable $\mathfrak{A}_2$ omitting p_0 and p_1. Proceeding inductively we get an infinite sequence of decidable models.

4. Theories with few models

Throughout this section T is a complete decidable theory.

If T is $\aleph_0$-categorical then by Theorem 2.1 its unique countable model is decidable.

Harrington [2] proves that:

THEOREM 4.2. *If T is $\aleph_1$-categorical then all of its countable models are decidable.*

The following question was first raised by A. Nerode.

If T has only a finite number of countable models, are they all decidable?

Notice that by Theorem 3.4 the prime model will be decidable.

I answered this question by giving an example of a theory with six countable models of which only the prime one was decidable. Later, Peretyat'kin [4] gave an example of a theory with three models of which only the prime one was decidable. By a theorem of Vaught [5] no complete theory has exactly two countable models. The following example of the six model case is due to Lachlan.

EXAMPLE. We use the following result of recursion theory. There is a recursive ordering, L, of the natural numbers which has order type $\omega + \omega^*$ but such that the set $\{n;$ there are only a finite number of m with $mLn\}$ is not recursive.

The theory T involves two binary relations E and $<$ and the axioms say:

i) E is an equivalence relation,

ii) for each $0 < n < \omega$ there is exactly one equivalence class, say c_n, with exactly n elements,

iii) the equivalence classes are linearly ordered by $<$ and have order type η (the rationals),

iv) $c_n < c_m$ whenever nLm.

The isomorphism type of a countable model is determined by the order type of those equivalence classes with an infinite number of c_n's both above and below them. There are six possibilities: 1, η, $\eta + 1$, $1 + \eta$, $1 + \eta + 1$, and $\varnothing$. Only the last of these has a decidable model for otherwise one could recursively determine the first half of the ordering L.

In the above example and in the one due to Peretyat'kin the reason not all countable models are decidable is that there are non-recursive types. In the next section we consider the effect of assuming all types recursive.

5. Consider the following four assumptions about a complete theory T:

i) T is decidable.

ii) Every type of finite sequence consistent with T is recursive.

iii) There is a recursive list of the types consistent with T.

iv) There is a decidable model realizing all types consistent with T.

It is clear that iv) implies iii) implies ii) implies i). The example in the last section is a theory satisfying i) but not ii). We shall next give an example of a theory satisfying ii) but not iii). To do this we shall use the following fact of recursion theory.

There is a closed set $Q \subseteq 2^\omega$ such that 1) the set of non-empty neighborhoods is recursive, 2) every point $p \in Q$ is recursive, but 3) there is no recursive list of the points in Q.

The theory T is now a monadic theory with monadic relations $R_n (n \in \omega)$ such that the consistent 1 types correspond to Q.

In the next section we shall show that iii) implies iv).

6. Saturated models

The main theory of this paper is

THEOREM 6.1. *If there is a recursive list of all the finite types consistent with T then T has a decidable saturated model.*

Note that the converse to the theorem is trivial.

PROOF. Assume L is the recursive list of types. There is no loss of generality in assuming each type appears infinitely often. The proof is a modification of the Henkin construction involving a priority argument. So let c_n and σ_n $(n \in \omega)$ be as in Section 2. We shall define (by induction on t) sentences $\theta(t)$ where $\theta(0)$ is a tautology, $\theta(2n+1) = \theta(2n+2)$ and $\theta(2n+1)$ is either $\theta(2n)^\wedge\sigma(n)$ or $\theta(2n)^\wedge \neg\sigma(n)$.

Let $n = \langle J(n), \mathrm{K}(n)\rangle$ be some standard enumeration of pairs of natural numbers.

Auxiliary to our definition of θ, we shall define, by induction on t, functions $f(n,t)$, $g(n,t)$, $\mathrm{X}(n,t)$ and $R(n,t)$. These functions will have "undefined" among their possible values and we shall say "$f(n)$ is undefined at time t" to mean $f(n,t) =$ undefined.

At time t, $f(n)$ will be defined on some finite initial segment of ω, $g(n)$ on an initial segment of length one longer. The values of X at time t depend on those of f and g as follows:

$$\mathrm{X}(n) = \langle c_0, \cdots, c_{m-1}, c_{f(0)}, \cdots, c_{t(n-1)}\rangle$$

where $m = \max\{J(n'); n' < n\}$.

Note that $X(n)$ is defined when $g(n)$ is and its length is independent of t. The values of R are requirements on θ thus:

$R(2n) = \theta$ is consistent with $\mathrm{X}(n)$ having type $L(g(n))$,

$R(2n+1) \equiv$ either θ is consistent with $\langle c_0, \cdots, c_{J(n)-1}, c_{f(n)}\rangle$ having type $L(\mathrm{K}(n))$ or θ implies that for no x does $\langle c_0, \cdots, c_{J(n)-1}, x\rangle$ have type $L(K(n))$.

Notice that $R(2n)$ and $R(2n+1)$ are defined when $g(n)$ and $f(n)$ respectively are defined.

The inductive definitions of $\theta(t)$ of $f(n,t)$ and $g(n,t)$ are as follows.

$t = 0$ $f(n)$ undefined for all n

$L(g(0))$ an index for T

$g(n)$ undefined for $n \neq 0$

$\theta(0)$ a tautology

$t = 2n+2$ $\theta(t) = \theta(t-1)$

Let $n_0 =$ least n with $f(n, t-1)$ undefined $f(n,t) = f(n, t-1)$ $n \neq n_0$ $f(n_0, t) =$ least r such that c_r does not occur in $\theta(t)$ and for all m and all $t' < t$, $f(m, t') \neq r$,

$$g(n,t) = g(n, t-1) \qquad n \neq n_0 + 1.$$

For $n = n_0 + 1$ there are two cases:
Case A) $f(n_0, t') =$ undefined for all $t' < t$ then $g(n_0 + 1, t) =$ least r such that $\theta(t)$ satisfies $R(2n)$.
Case B) otherwise. Let $s_0 =$ last preceding value of $f(n_0)$ and t_0 least t' with $f(n_0, t') = s_0$. Then $g(n_0 + 1, t) =$ least $r > g(n_0 + 1, t_0)$ such that $\theta(t)$ satisfies $R(2n)$.

$t = 2n + 1$

For each m if one assumes that $\theta(t-1)$ satisfies $R(m, t-1)$ then it is possible to choose σ_n or $\neg \sigma_n$ so that $\theta(t)$ satisfies $R(m, t-1)$. (Undefined requirements are assumed satisfied.) However, different m's may require different choices. Choose $\theta(t)$ to satisfy longest possible initial segment.
Case I. All requirement satisfied. $f(m,t) = f(m, t-1)$ and $g(m,t) = g(m, t-1)$ for all m.
Case II. First requirement violated is $R(2m+1)$. Then $f(m', t) = f(m', t-1)$ for $m' < m$ and $f(m', t) =$ undefined for $m' \geqq m$, $g(m', 1) = g(m', t-1)$ for $m' \leqq m$ and $g(m', t) =$ undefined for $m' > m$.
Case III. First requirement violated is $R(2m)$. Increase value of $g(m)$ until $R(2m)$ satisfied. Continue until reach Case I or II.

It will be left to the reader to verify that:

1) Since L is a recursive list the above inductions define recursive functions.

2) For each n there is a t_n such that $f(n)$ and $g(n)$ are defined and independent of t for all $t > t_n$.

3) Hence by the requirements $R(2n)$ $\{\theta(t); t \in \omega\}$ defines the complete diagram of a saturated model.

COROLLARY 6.2. *If every finite type consistent with T is recursive and the countable saturated model is not decidable then there are an infinite number of non-isomorphic decidable models of T.*

PROOF. Every recursive type is realized in some decidable model. If there were a finite number of decidable models realizing them all then there would be a recursive list of all the types.

7. Other results and questions

Vaught [5] shows that no complete theory has exactly two countable models. Millar [3] gives an example to show

THEOREM 7.1. *There is a complete theory with exactly two decidable models.*

A natural question is to find conditions under which Vaught's result applies to decidable models. One such is:

THEOREM 7.2. *Suppose T is complete, not $\aleph_0$-categorical and every type consistent with T is recursive. Then T has at least three non-isomorphic decidable models.*

PROOF. By Theorems 3.4 and 6.2 if T has only a finite number of decidable models then both the prime and saturated models are decidable. Now, imitating Vaught's argument we can construct a model realizing a non-principal type but omitting some non-principal extension of that type.

This leads to a modification of Nerode's question.

QUESTION. Suppose T has exactly $n < \omega$ countable models and every type consistent with T is recursive.

1) Is every countable model of T decidable?

2) If not, what is the least n giving a counterexample?

Notice that by 7.2 the answer to 2) is not three.

Another question is whether the results for prime and saturated models can be extended to homogeneous models in general.

QUESTION. Suppose there is a recursive list of the types appearing in a countable homogeneous model. Is the model necessarily decidable?

Millar [3] shows that

THEOREM 7.3. *Suppose $\mathfrak{A}$ is a countable homogeneous model, L_1 a Σ_2^0 list of the types appearing in $\mathfrak{A}$ and L_2 a Σ_2^0 list of those* recursive *types omitted in $\mathfrak{A}$. Then $\mathfrak{A}$ is decidable.*

Notice that even for the saturated model this strengthens our Theorem 6.1.

One final counterexample.

THEOREM 7.4. *There is a decidable T with a recursive list of the* recursive *types consistent with T but no decidable model of T is recursively saturated in the sense of Barwise and Schipf* [1].

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